International Journal of Theoretical Physics, Vol. 22, No. 6, 1983

Vertex Calculation for a Composite Particle

Andrea Raspini

Dept. of Physics, University of Lowell, Lowell, MA. 01854[1]

Received June 24, 1981

The classical treatment and the quantization of composite relativistic systems is given a manifestly covariant formulation in presence of constraints. A particular formulation of Feynman's quantum mechanics is used to treat the scattering of composite relativistic systems. A covariant harmonic oscillator model is employed to calculate vertices of interactions: the results are similar to the corresponding ones in the usual field theories, but the presence of some convergence factors gives hope that a theory with composite particles may be finite.

1. INTRODUCTION

This paper calculates the vertex function for the scattering of composite particles, employing a model of covariant massive harmonic oscillator (Kim and Noz, 1973). This model is reviewed in Section 4, where the general result for the vertex calculation is presented too (for all the possible states of integer spin of interacting particles).

In Sections 2 and 3 the classical and quantum treatment of relativistic discrete systems is revised starting from Dirac's method, i.e., in presence of constraints (Dirac, 1964). A particular formulation of Feynman's quantum mechanics principle (Feynman and Hibbs, 1965) is proposed, in order to take into account constrained problems. The geometrical scattering formalism (Mandelstam, 1973) is summarized, with the aim of presenting it for the vertex calculation.

In Section 5 the vertex function is examined in two particular cases (scattering of three scalars and scattering of a spin-1 boson and a scalar into a scalar). The results are similar to those of usual field theories, but some extra factors look like convergence factors for large momenta.

[1]Work supported, in part, by INFN and Fondazione A. Della Riccia, Florence, Italy.

0020-7748/83/0600-0485$03.00/0

2. CLASSICAL TREATMENT OF A DISCRETE RELATIVISTIC SYSTEM

I will consider a relativistic system S described by N generalized coordinates. They will be called $q^1 \cdots q^N$ and the evolution parameter will be called τ.

It is well known that the corresponding Lagrangian cannot depend explicitly on τ and has to be homogeneous of degree one in the generalized velocities

$$\dot{q}^i = \frac{dq^i}{d\tau}, \qquad i = 1 \cdots N \tag{1}$$

in order to be invariant under reparametrization (Bolza, 1904).

If $\dot{q}$ means the whole set of velocities and q means the whole set of coordinates, calling $L(\dot{q}, q)$ the Lagrangian we are talking about, the usual definition of canonical momenta is

$$p_i(\dot{q}, q) = -\frac{\partial L(\dot{q}, q)}{\partial \dot{q}^i} \tag{2}$$

from which follows the usual definition of the Hamiltonian:

$$H(\dot{q}, q) = -p_i \dot{q}^i - L \tag{3}$$

where the summation convention of repeated up and down indices is used.

Because of the homogeneity of L, H is identically the zero function:

$$H(\dot{q}, q) \equiv 0 \tag{4}$$

But at the same time the same homogeneity property implies

$$\det\left[\frac{\partial^2 L(\dot{q}, q)}{\partial \dot{q}^i \partial \dot{q}^k}\right] \equiv 0 \tag{5}$$

and this condition clearly forbids a regular Lagrangian or Hamiltonian treatment of the problem, since the Lagrangian is of the kind called singular (see, for the argument, Sudarshan and Mukunda, 1974, and Dirac, 1964).

A direct Hamiltonian approach for singular problems is Dirac's method (Dirac, 1964), which I will shortly describe, in a revised version.

The Lagrangian $L(\dot{q}, q)$ can be used for all the values of $\dot{q}, q$ for which it is mathematically regular and suitable to the problem, say $(\dot{q}, q) \in Q \subset$

$\mathbb{R}^{2N}$. But in different subsets of Q the rank of the Hessian matrix

$$(W_{ik}) = \left(\frac{\partial^2 L(\dot{q}, q)}{\partial \dot{q}^i \partial \dot{q}^k} \right) \tag{6}$$

may be different, and since separate treatments are to be done for different ranks (which correspond to different physical situations), we now consider the total $2N$-dimensional volume $Q_R \subset Q$, for which

$$\operatorname{rank}(W_{ik}) \equiv R < N \tag{7}$$

We cannot express all the velocities $\dot{q}$'s as functions of the canonical variables p's, q's and, on the contrary, we find $(N-R)$ relations, called primary constraints, of the kind

$$\mathring{\Omega}_m(p, q) = 0, \qquad m = 1 \cdots N - R \tag{8}$$

directly deducible from the definitions of the p's and not involving any of the $\dot{q}$'s (Shanmugadhasan, 1973). Then, according to the theory of constrained variational calculus, using that first set of constraints the usual Hamiltonian principle is to be replaced by the following one (remember $H \equiv 0$):

$$\delta \int_a^b [-p_i(\tau)\dot{q}^i(\tau) + \mathring{\lambda}^m(\tau)\mathring{\Omega}_m(\tau)]\, d\tau = 0 \tag{9}$$

applied in the region described by all the restrictive relations already introduced, where $\mathring{\lambda}$'s are arbitrary functions of the parameter τ. We get the set of equations (forget, at first reading, the $\sim$ underlining)

$$\dot{p}_i \underset{\sim}{\equiv} \mathring{\lambda}^m \frac{\partial \mathring{\Omega}_m(p, q)}{\partial q^i}$$

$$\dot{q}^i \underset{\sim}{\equiv} -\mathring{\lambda}^m \frac{\partial \mathring{\Omega}_m(p, q)}{\partial p_i} \tag{10}$$

completed, as forewarned, by the restrictions

$$\mathring{\Omega}_m \underset{\sim}{\equiv} 0, \qquad (\dot{q}, q) \underset{\sim}{\in} Q_R \tag{11}$$

the last one of which, just using (10), can be expressed as follows:

$$(p, q, \mathring{\lambda}^m) \underset{\sim}{\in} \mathcal{F}_R \subset \mathbb{R}^{2N+(N-R)} \qquad (m = 1 \cdots N - R) \tag{12}$$

To properly stress the presence of restrictions we have used relation symbols underlined with a $\sim$. Each underlined symbol acquires a "weak value," which means it is to be considered valid only in the presence of all the restrictive conditions already defined (Sudarshan and Mukunda, 1974).

Now, since the only requirement for correct dynamics is having restrictions which are constants of the motion (Dirac, 1964), we have to find an additional set of conditions so that the complete set of restrictions can be self-consistent in maintaining itself during the flow of the "time" τ; principle (9) is updated for each new constraint introduced, i.e., the Hamiltonian equations of motion are, step by step, updated too.

Considering therefore the evolution of constraint conditions (having called "constraint" a restriction written in the form of an equality), we can write [see equations (10)]

$$\begin{aligned} \frac{d\mathring{\Omega}_m}{d\tau} &= \frac{\partial \mathring{\Omega}_m}{\partial q^i}\dot{q}^i + \frac{\partial \mathring{\Omega}_m}{\partial p_i}\dot{p}_i \\ &\underset{\sim}{=} -\mathring{\lambda}^{m'}\frac{\partial \mathring{\Omega}_m}{\partial q^i}\frac{\partial \mathring{\Omega}_{m'}}{\partial p_i} + \mathring{\lambda}^{m'}\frac{\partial \mathring{\Omega}_m}{\partial p_i}\frac{\partial \mathring{\Omega}_{m'}}{\partial q^i} \end{aligned} \tag{13}$$

and so we have to require

$$\mathring{\lambda}^{m'}\{\mathring{\Omega}_m, \mathring{\Omega}_{m'}\} \underset{\sim}{=} 0 \tag{14}$$

In the region (11), which we are considering, the rank of the matrix

$$(\mathring{\Omega}_{mm'}) = (\{\mathring{\Omega}_m, \mathring{\Omega}_{m'}\}) \tag{15}$$

will take its minimum B $(0 \leqslant B \leqslant N - R)$ in a certain subregion of the type

$$\mathring{\Omega}_m \underset{\sim}{=} 0, \qquad (p, q, \mathring{\lambda}^m) \underset{\sim}{\in} \mathcal{F}_R, \qquad (p,q) \underset{\sim}{\in} F_B \subset \mathbb{R}^{2N} \tag{16}$$

being major in all other subregions.

The region included in F_B can be described by a maximum number of constraint relations (secondary constraints)

$$\mathring{\chi}_a \underset{\sim}{=} 0, \qquad a = 1 \cdots A_B \tag{17}$$

and by a possible supplementary condition not allowed to be expressed in constraint form [say, $(p,q) \underset{\sim}{\in} F'_B \subset F_B$]. That means we can write, to express the region (16),

$$\mathring{\Omega}_m \approx 0, \qquad \mathring{\chi}_a \approx 0, \qquad (p, q, \mathring{\lambda}^m) \in \tilde{\mathscr{F}}_{R,B} \subset \mathscr{F}_R$$
$$m = 1 \cdots N - R, \qquad a = 1 \cdots A_B \tag{18}$$

having formally combined the supplementary condition $(p, q) \in F'_B$ with restriction (12), updating it.

The area (18) represents an updated set of restrictions: in other words, during the process of updating the set of restrictions, we are leaving the $\mathring{\lambda}$'s as undetermined as possible. (It is intended that the residual restriction not expressed in the form of constraint remains so little restrictive on $\mathring{\lambda}$'s that we can forget its conditioning on them.)

At this point, we introduce (via some new $\mathring{\lambda}$'s) the new constraints in the principle (9) and derive from it the new Hamiltonian equations:

$$\begin{aligned} \dot{p}_i &\approx \mathring{\lambda}^h \frac{\partial \mathring{V}_h(p,q)}{\partial q^i} \\ \dot{q}^i &\approx -\mathring{\lambda}^h \frac{\partial \mathring{V}_h(p,q)}{\partial p_i} \end{aligned} \qquad (h = 1 \cdots N - R + A_B)$$

$$\begin{aligned} \mathring{V}_h &= \mathring{\Omega}_h && \text{if } h = 1 \cdots N - R \\ \mathring{V}_h &= \mathring{\chi}_{h-(N-R)} && \text{if } h = N - R + 1 \cdots N - R + A_B \end{aligned} \tag{19}$$

The new equations for the motion constancy of constraints are

$$\mathring{\lambda}^{h'} \{ \mathring{V}_h, \mathring{V}_{h'} \} \approx 0, \qquad h, h' = 1 \cdots N - R + A_B \tag{20}$$

and again the possible $\mathring{\lambda}$'s that guarantee them depend on the rank of the matrix $(\mathring{V}_{hh'}) = (\{\mathring{V}_h, \mathring{V}_{h'}\})$.

If that rank is constant in the area

$$\mathring{V}_h \approx 0, \qquad (p, q, \mathring{\lambda}^h) \in \mathscr{F}_{R,B}, \qquad h = 1 \cdots N - R + A_B \tag{18$'$}$$

[where the last condition corresponds to the last one in (18), but with $(\dot{q}, q) \in Q_R$ developed via the new Hamiltonian equations (19)], we have no reason to divide the region again. [In the same way if the rank of matrix (15) was constant in the region (11) we would not introduce any secondary constraints.] But if that rank is variable it will take its minimum B'

$(B \leqslant B' \leqslant N - R + A_B)$ in a region

$$\mathring{\Omega}_m \approx 0, \qquad \mathring{\chi}_a \approx 0, \qquad a = 1 \cdots A_{B'} \geqslant A_B$$
$$(p, q, \mathring{\lambda}^h) \underset{\approx}{\in} \tilde{\mathcal{F}}_{R,B,B'} \subset \mathcal{F}_{R,B} \qquad (h = 1 \cdots N - R + A_B) \tag{21}$$

The method is iterated until we find a region

$$\mathring{\Omega}_m \approx 0, \qquad \mathring{\chi}_a \approx 0, \qquad a = 1 \cdots A_{B^{(n)}}$$
$$(p, q, \mathring{\lambda}^h) \underset{\approx}{\in} \mathcal{F}_{R,B,B'\cdots B^{(n)}} \qquad (h = 1 \cdots N - R + A_{B^{(n)}}) \tag{22}$$

in which it is impossible, following the described procedure, to introduce new restrictions. This region, also written as follows:

$$\mathring{V}_h \approx 0, \qquad (p, q, \mathring{\lambda}^h) \underset{\approx}{\in} \mathcal{F}_{R,B,B'\cdots B^{(n)}}$$
$$\mathring{V}_h = \mathring{\Omega}_h \qquad \text{if } h = 1 \cdots N - R$$
$$\mathring{V}_h = \mathring{\chi}_{h-(N-R)} \qquad \text{if } h = N - R + 1 \cdots N - R + A_{B^{(n)}} \tag{23}$$

represents the complete set of restrictions to the Hamiltonian equations which now are

$$\dot{p}_i \approx \mathring{\lambda}^h \frac{\partial \mathring{V}_h(p, q)}{\partial q^i}$$
$$\dot{q}^i \approx -\mathring{\lambda}^h \frac{\partial \mathring{V}_h(p, q)}{\partial p_i} \qquad (h = 1 \cdots N - R + A_{B^{(n)}}) \tag{24}$$

having started from a rank R of the Hessian matrix (6) and assuming, as we will make precise, that the residual restriction not expressed in the form of a constraint does not need any generation of further conditions.

The corresponding equations for constancy of constraints are

$$\mathring{\lambda}^{h'}\{\mathring{V}_h, \mathring{V}_{h'}\} \approx 0, \qquad h, h' = 1 \cdots N - R + A_{B^{(n)}} \tag{25}$$

and it is clear they condition only $B^{(n)}$ of the $(N - R + A_{B^{(n)}})$ degrees of freedom of the $\mathring{\lambda}$ set. (As usual the residual condition is considered as not affecting the determination of $\mathring{\lambda}$'s.)

A rearrangement of constraints can be done in order to eliminate $B^{(n)}$ of the $\mathring{\lambda}$'s determining them as zero functions (Sudarshan and Mukunda, 1974). The treatment we have developed maintains its validity if we change the $\mathring{V}$ functions into the new ones

$$V_h(p,q)=S_h^{h'}(p,q)\mathring{V}_{h'}(p,q) \tag{26}$$

where $(S_h^{h'})$ is a $(N-R+A_{B^{(n)}})\times(N-R+A_{B^{(n)}})$ weakly nonsingular matrix.

We will, from now on, use the new symbols V_h to mean we have rendered maximum the number of constraints which are weakly represented by first-class functions.[2] In this way, rewriting the (25) system as follows:

$$\lambda^{h'}\{V_h, V_{h'}\}\approx 0 \tag{27}$$

a maximum number of equations and terms of equations disappears and the remaining ones

$$\lambda^{h'_2}\{V_{h_2}, V_{h'_2}\}\approx 0 \tag{28}$$

which contain $B^{(n)}\lambda$'s and have (Sudarshan and Mukunda, 1974)

$$\operatorname{rank}\left(V_{h_2h'_2}\right)\underset{\sim}{\equiv} B^{(n)} \qquad \left[\text{i.e., } \det\left(V_{h_2h'_2}\right)\underset{\sim}{\neq} 0\right] \tag{29}$$

give clearly

$$\lambda^{h_2}\equiv 0 \tag{30}$$

At this point the Hamiltonian equations are

$$\dot{p}_i\approx\lambda^{h_1}\frac{\partial V_{h_1}}{\partial q^i}, \qquad \dot{q}^i\approx-\lambda^{h_1}\frac{\partial V_{h_1}}{\partial p_i} \tag{31}$$

where the h_1 index runs through $(N-R+A_{B^{(n)}}-B^{(n)})$ values and the λ^{h_1}'s remain undetermined. The presence of undetermined quantities means the presence of invariances in the theory: the choice of particular gauges on them can give determination to the λ^{h_1}'s. (For instance an invariance that is always present is the reparameterization one; by itself it gives rise to one undetermined λ whose value can be chosen fixing the physical meaning of τ.)

[2]We remember the meaning of this: if $V_h(p,q)\approx 0$ $(h=1\cdots)$ is a set of equations, a function V_{h_0} of V's is weakly "first class" if $\{V_{h_0}, V_h\}\approx 0\ \forall h$.

The residual condition $(p, q, \lambda^h) \lessapprox \mathcal{F}_{R, B, B' \cdots B^{(n)}}$ that we left in this way, we assume is now rewritten as follows:

$$(p, q, \lambda^{h_1}) \lessapprox \bar{\bar{\mathcal{F}}}_{R, B, B' \cdots B^{(n)}} \subset \mathbb{R}^M \tag{32}$$

where

$$M = 2N + (N - R) + A_{B^{(n)}} - B^{(n)}$$

and we also assume it is a constant of the motion if we choose properly a $\bar{\lambda}^{h_1}$ set for the λ^{h_1}'s.

At last we define

$$H_T(p, q, \bar{\lambda}^{h_1}) = \bar{\lambda}^{h_1} V_{h_1}(p, q) \tag{33}$$

and we can write the complete restricted Hamiltonian problem in the following form:

$$\dot{p}_i \approxeq \{p_i, H_T\}, \qquad \dot{q}^i \approxeq \{q^i, H_T\}$$

$$V_h \approxeq 0, \qquad (p, q, \bar{\lambda}^{h_1}) \lessapprox \bar{\bar{\mathcal{F}}}_{R, B, B' \cdots B^{(n)}} \tag{34}$$

or, in general, for any function of the motion $A(p, q)$ we have

$$\dot{A}(p, q) \approxeq \{A, H_T\} \tag{35}$$

After having established a Poisson formalism[3] via the "total Hamiltonian" H_T, we now define a "total Lagrangian" L_T generated by H_T.

We consider N regular functions $g_i(\dot{q}, q, \bar{\lambda}^{h_1})$ and define

$$\mathcal{K}^i(\dot{q}, q, \bar{\lambda}^{h_1}) = -\frac{\partial H_T(g(\dot{q}, q, \bar{\lambda}^{h_1}), q, \bar{\lambda}^{h_1})}{\partial g_i} \tag{36}$$

$$\mathcal{R} = \{(\dot{q}, q, \bar{\lambda}^{h_1}) : (g, q, \bar{\lambda}^{h_1}) \in \bar{\bar{\mathcal{F}}}_{R, B, B' \cdots B^{(n)}}\} \tag{37}$$

If it is possible to satisfy

$$\det\left(\frac{\partial g_i}{\partial \dot{q}^k}\right) \neq 0 \qquad \text{at least in } \mathcal{R} \tag{38}$$

[3]A Dirac formalism can be originated, which modifies Poisson brackets into "Dirac brackets": see Dirac (1964) or Sudarshan and Mukunda (1974). Note, anyway, that the Dirac bracket formalism does not keep the manifest covariance.

$$\mathcal{K}^i(\dot{q}, q, \bar{\lambda}^{h_1}) = \dot{q}^i \qquad \text{at least in } \mathfrak{R} \tag{39}$$

then the function

$$L_T(\dot{q}, q, \bar{\lambda}^{h_1}) = -g_i(\dot{q}, q, \bar{\lambda}^{h_1})\dot{q}^i - H_T(g(\dot{q}, q, \bar{\lambda}^{h_1}), q, \bar{\lambda}^{h_1}) \tag{40}$$

is mathematically regular and nonsingular at least in $\mathfrak{R}$; besides, considered as an usual Lagrangian in $\mathfrak{R}$ ($\bar{\lambda}^{h_1}$ taken as parameters), it gives the Hamiltonian problem

$$\dot{\tilde{p}}_i = \frac{\partial \tilde{H}(\tilde{p}, q, \bar{\lambda}^{h_1})}{\partial q^i}, \qquad \dot{q}^i = -\frac{\partial \tilde{H}(\tilde{p}, q, \bar{\lambda}^{h_1})}{\partial \tilde{p}_i}$$

$$(\tilde{p}, q, \bar{\lambda}^{h_1}) \in \bar{\bar{\mathfrak{F}}}_{R, B, B' \cdots B^{(n)}} \tag{41}$$

where we have called $\tilde{p}$'s and $\tilde{H}$ the canonical momenta and the usual Hamiltonian derived from L_T and where it is easy to see

$$\tilde{H} \equiv H_T \tag{42}$$

(and so also $\tilde{p} \equiv p$).

That means L_T is the Lagrangian which generates, with usual nonsingular treatment, the Hamiltonian problem without constraints. If $\mathfrak{R}_{\max} \supset \mathfrak{R}$ is the maximum set in which L_T is definable with the above procedure, starting from L_T on $\mathfrak{R}_{\max}$ we can find the Hamiltonian problem deprived of a maximum part of the residual restriction too.

In the following I will use L_T for a particular purpose, while quantizing what has been established; the quantization will be done starting from Poisson brackets.[4]

3. QUANTIZATION AND SCATTERING

For the following we will assume that the residual condition in (34) can be written as follows:

$$G_r(p, q, \bar{\lambda}^{h_1})\, \underset{\sim}{\rho_r}\, 0, \qquad r = 1 \cdots D \tag{43}$$

where each ρ_r is a relation symbol belonging to the set $>, \geqslant, <, \leqslant, \neq$.

Considering now the usual quantization procedure $\{,\} \to -i[,]$, in the Schrödinger picture the canonical variables p's and q's become linear

[4] For a Dirac bracket quantization see Dirac (1964).

Hermitian operators $\hat{p}$'s, $\hat{q}$'s not depending on τ and characterized by a commutator $[\hat{p}_j, \hat{q}^k] = i\delta_j^k$ (we use $\hbar = c = 1$).

The problem (34), if we leave apart, for a while, the non-first-class constraint functions, could be quantized as follows (Dirac, 1964):

$$
\begin{aligned}
&i\frac{d}{d\tau}|\Phi(\tau)\rangle = \hat{H}_T(\tau)|\Phi(\tau)\rangle \\
&\hat{V}_{h_1}|\Phi(\tau)\rangle = 0, \qquad h_1 = 1 \cdots N - R + A_{B^{(n)}} - B^{(n)} \\
&\langle\Phi(\tau)|\hat{G}_r(\tau)|\Phi(\tau)\rangle \underset{\sim}{\rho_r} 0, \qquad r = 1 \cdots D
\end{aligned} \tag{44}
$$

where $\hat{V}_{h_1}$ is the linear Hermitian operator (not depending on τ) corresponding to a constraint function V_{h_1}; $\hat{H}_T(\tau)$ and $\hat{G}_r(\tau)$ are the linear Hermitian operators corresponding to H_T and G_r, and, in case, depending on τ only through the $\bar{\lambda}^{h_1}(\tau)$'s (which remain classical multipliers). Notice that the first-class constraint functions V_{h_1}'s meet (Dirac, 1964)

$$\{V_{h_1}(p,q), V_{h'_1}(p,q)\} = \alpha^{h''_1}{}_{h_1 h'_1}(p,q) V_{h''_1}(p,q) \tag{45}$$

and so we can also have[5]

$$\left[\hat{V}_{h_1}, \hat{V}_{h'_1}\right] = i\hat{\alpha}^{h''_1}{}_{h_1 h'_1}\hat{V}_{h''_1} \tag{46}$$

Of course that is the reason the quantum problem (44) looks consistent, remembering that H_T too is made up of first-class constraint functions. Actually this last circumstance means that the Schrödinger equation in (44), in consequence of the "observations" of $\hat{V}_{h_1}$ operators, is simply reduced to this:

$$i\frac{d}{d\tau}|\Phi(\tau)\rangle = 0 \tag{47}$$

At this point, for a further analysis including the non-first-class constraint functions [whose number $B^{(n)}$ is even, as stated by $\det(V_{h_2 h'_2}) \underset{\sim}{\neq} 0$], we need the whole set of constraints to be in a "standard" form, which we will make precise. Direct "observation" of non-first-class constraints is forbidden, but, in the standard form, there is a way of "combining" them properly in a complex function formalism.

[5]We are assuming there are no problems in the correspondence rule $\{,\} \to -i[,]$; for a discussion of this, see, for instance, Von Neumann (1955) and Rosen (1969).

It is generally possible to build up the whole set of constraints in such a way that it satisfies[6] (45) and, calling Q_{i_2}, P_{i_2}: $i_2 = 1 \cdots H = B^{(n)}/2$ the non-first-class constraint functions:

$$\begin{aligned}
&\{Q_{i_2}, Q_{i'_2}\} = \{P_{i_2}, P_{i'_2}\} \equiv 0, \qquad i_2, i'_2 = 1 \cdots H \\
&\{Q_{i_2}, P_{i'_2}\} \equiv 0 \qquad \text{if } i_2 \neq i'_2 \\
&\{Q_{i_2}, P_{i_2}\} \gtrsim 0, \qquad i_2 = 1 \cdots H
\end{aligned} \tag{48}$$

and furthermore

$$\{V_{h_1}, T_{i_2}\} = \beta^{h'_1}_{h_1 i_2} V_{h'_1} + \gamma^{i'_2}_{h_1 i_2} T_{i'_2} \tag{48'}$$

where each T_{i_2} is a complex linear combination of the following kind:

$$T_{i_2} = Q_{i_2} + \mathrm{i} P_{i_2}, \tag{49}$$

Of course T_{i_2}'s, which are the only complex classical functions introduced until now, have the property

$$T_{i_2} = 0 \Leftrightarrow Q_{i_2} = P_{i_2} = 0$$

and for the corresponding linear operators there should be no problem of observation[5]:

$$\hat{T}_{i_2} |\Phi(\tau)\rangle = 0, \qquad i_2 = 1 \cdots H \tag{50}$$

So, for the $\hat{T}_{i_2}$ are annihilation-like operators, it looks reasonable [considering the (50) as auxiliary conditions] to assume that any physical state is determined by these equations, including "observations" of both kinds of constraints:

$$\begin{aligned}
&\hat{V}_{h_1} |\Phi(\tau)\rangle = 0, \qquad \hat{T}_{i_2} |\Phi(\tau)\rangle = 0 \\
&\langle \Phi(\tau) | \hat{G}_r(\tau) | \Phi(\tau)\rangle \ \rho_r\ 0, \qquad \mathrm{i}\frac{d}{d\tau} |\Phi(\tau)\rangle = 0
\end{aligned} \tag{51}$$

The Schrödinger equation

$$\mathrm{i}\frac{d}{d\tau} |\Phi(\tau)\rangle = \hat{H}_T(\tau) |\Phi(\tau)\rangle \tag{52}$$

[6]What follows is a consequence of Theorem VII, 24 in Schouten and Kulk (1949).

does not completely lose its meaning in view of (47); in fact we can consider that any ket $|\Phi(\tau)\rangle$ solution of (52) is a generic representative state of our system, generally not physical but virtual, suitable, in this case, as a propagator line of a Feynman graph. So it is interesting to look at the evolution of kets satisfying simply (52) (representative kets).

If $|\Phi(\tau_0)\rangle$ is a representative ket at a "time" τ_0 it is well known that we can get a unitary operator $\hat{U}_\tau(\tau_0)$ which gives the evolution of $|\Phi(\tau_0)\rangle$:

$$|\Phi(\tau)\rangle = \hat{U}_\tau(\tau_0)|\Phi(\tau_0)\rangle \tag{53}$$

It is well known too that for ordinary nonsingular problems there exists the Feynman expression (Feynman and Hibbs, 1965)

$$\langle q''|\hat{U}_\tau(\tau_0)|q'\rangle = \int_{\substack{q(\tau_0)=q' \\ q(\tau)=q''}} \exp\left[\mathrm{i}\int_{\tau_0}^{\tau} dt\, L(\dot{q}(t), q(t))\right] \mathfrak{D}(q(t)) \tag{54}$$

where L is the classical Lagrangian of the problem and $|q\rangle$ is a basis of eigenkets of $\hat{q}$ not depending on τ and generating the Hilbertian space of representative kets.

The formula (54) can be modified for singular problems. If $L_T(\dot{q}, q, \bar{\lambda}^{h_1})$ is the total Lagrangian and the $\mathfrak{R}_{\max}$ set we defined in the last part of Section 2 is the whole space $\mathbb{R}^M$ [see formula (32) for evidence], it is quite obvious that the Schrödinger equation (52) has the same relation with L_T that exists between the Lagrangian L of a classical nonsingular problem and the corresponding Schrödinger equation.

That means we can write

$$\langle q''|\hat{U}_\tau(\tau_0)|q'\rangle = \int_{\substack{q(\tau_0)=q' \\ q(\tau)=q''}} \exp\left[\mathrm{i}\int_{\tau_0}^{\tau} dt\, L_T(\dot{q}(t), q(t), \bar{\lambda}^{h_1}(t))\right] \mathfrak{D}(q(t)) \tag{55}$$

which is the Feynman principle adapted for singular problems.

Of course it is possible to describe a path integral geometrical scattering formulation too (Mandelstam, 1973). Consider the graph in Figure 1 with the following definitions:

(i) $|\Phi_{(1)}\rangle, |\Phi_{(2)}\rangle, |\Phi_{(3)}\rangle, |\Phi_{(4)}\rangle$ and the line [0] are "free" states, in the meaning that their relative total Hamiltonians $\hat{H}_{T(1)}(\tau_{(1)})$, $\hat{H}_{T(2)}(\tau_{(2)})$, $\hat{H}_{T(3)}(\tau_{(3)})$, $\hat{H}_{T(4)}(\tau_{(4)})$, $\hat{H}_{T(0)}(\tau_{(0)})$ do not contain anything concerning the interaction of Figure 1.

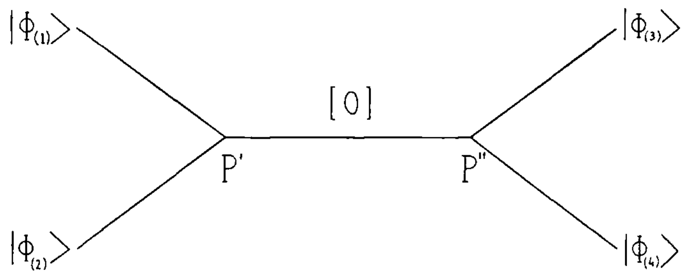

Fig. 1. The simplest scattering graph.

(ii) $|\Phi_{(1)}\rangle, |\Phi_{(2)}\rangle, |\Phi_{(3)}\rangle, |\Phi_{(4)}\rangle$ are fixed states of physical meaning (and so independent of their "time" parameters $\tau_{(1)}, \tau_{(2)}, \tau_{(3)}, \tau_{(4)}$).
(iii) [0] is a line of a virtual state.
(iv) τ', τ'' are the conventional values of $\tau_{(0)}$ at the "scattering points" P', P''.
(v) $|\rho_{(0)}\rangle$: $\rho_{(0)} \in \mathcal{S}$ is a set of kets (functions of an index $\rho_{(0)}$) not depending on $\tau_{(0)}$ and able to span the whole Hilbert space to which a state of [0] belongs.

Assuming, for convenience, that $\rho_{(0)}$ is a nondiscrete index, the geometric formulation of scattering is simply

$$\langle\langle \text{output} | \text{input} \rangle |_{\substack{\text{scattering} \\ \text{of Fig. 1}}}$$
$$= \int \langle \Phi_{(3)} \Phi_{(4)} | \rho''_{(0)} \rangle \Big|_{\substack{\text{interaction} \\ \text{at } P''}} \langle \rho''_{(0)} | \hat{U}_{(0)\tau''}(\tau') | \rho'_{(0)} \rangle$$
$$\times \langle \rho'_{(0)} | \Phi_{(1)} \Phi_{(2)} \rangle \big|_{\substack{\text{interaction} \\ \text{at } P'}} d\rho'_{(0)}\, d\rho''_{(0)}\, d(\tau'' - \tau') \tag{56}$$

Introducing the wave functions $\Phi_{(i)}$ (in $L_{(i)}$ coordinates $q_{(i)}$'s: $i = 1, 2$), and $F_{\rho_{(0)}} = \langle q_{(0)} | \rho_{(0)} \rangle$ (in $L_{(0)}$ coordinates $q_{(0)}$'s), the vertices in (56) can be expressed using this (Mandelstam, 1973):

$$| \ \langle \rho_{(0)} | \Phi_{(1)} \Phi_{(2)} \rangle \big|_{\substack{\text{interaction} \\ \text{at } P}} = C \int \left[F^{*}_{\rho(0)}(q_{(0)}) \Phi_{(1)}(q_{(1)}) \Phi_{(2)}(q_{(2)}) \right] \Bigg|_{\substack{q_{(0)} = f_P(\xi) \\ q_{(1)} = g_P(\xi) \\ q_{(2)} = h_P(\xi)}} d\xi \tag{57}$$

where C is a coupling constant and the connections

$$\begin{aligned} &q_{(0)}=f_P(\xi), \qquad q_{(1)}=g_P(\xi), \qquad q_{(2)}=h_P(\xi) \\ &q_{(0)}, f_P\in\mathbb{R}^{L_{(0)}}, \qquad q_{(1)}, g_P\in\mathbb{R}^{L_{(1)}} \\ &q_{(2)}, h_P\in\mathbb{R}^{L_{(2)}}, \qquad *\xi\in\mathbb{R}^{L_{(P)}} \end{aligned} \tag{58}$$

are considered able to define the "scattering point" P, whose dimensionality has been called $L_{(P)}$. Of course the propagator

$$\langle \rho''_{(0)}|\hat{U}_{(0)\tau''}(\tau')|\rho'_{(0)}\rangle \tag{59}$$

can be generally calculated directly, remembering the expression

$$\hat{U}_{(0)\tau''}(\tau')=\exp\left[-\mathrm{i}\int_{\tau'}^{\tau''} H_{T(0)}(\tau)\,d\tau\right] \tag{60}$$

In other cases, since we obviously have

$$\langle \rho''_{(0)}|\hat{U}_{(0)\tau''}(\tau')|\rho'_{(0)}\rangle=\int dq''_{(0)}\,dq'_{(0)}\langle \rho''_{(0)}|q''_{(0)}\rangle\langle q''_{(0)}|\hat{U}_{(0)\tau''}(\tau')|q'_{(0)}\rangle\langle q'_{(0)}|\rho'_{(0)}\rangle \tag{61}$$

we may use principle (55).

In conclusion of this section, we note that a propagator line [0] of a scattering graph should not be necessarily thought of as a generic representative state of the propagating system (i.e., a state simply satisfying the Schrödinger equation). It is quite clear that any quantum condition compatible with the Schrödinger equation may be used to restrict the set of representative states to a subset of virtual states suitable to the problem, i.e., states which correspond to the meaning of the propagation we are considering; the path integral quantum mechanics formulation here outlined is not affected by this.

4. INTERACTION VERTEX FOR PARTICLES DESCRIBED BY A COVARIANT MASSIVE HARMONIC OSCILLATOR MODEL

In this section I use a covariant massive harmonic oscillator model to compute the interaction vertex for particles described in that way. I first recapitulate the model (Kim and Noz, 1973; Karr, 1976; Kalb and Van Alstine, 1976).

In a pseudo-Euclidean frame of reference,[7] calling $\tilde{x}_1$ and $\tilde{x}_2$ the four-dimensional coordinates of the two oscillating points and defining the new variables

$$x=\tfrac{1}{2}(\tilde{x}_1+\tilde{x}_2), \qquad z=\tilde{x}_2-\tilde{x}_1 \tag{62}$$

the relativistic Lagrangian can be expressed as

$$L(\dot{x},\dot{z},x,z)=-K\left\{\left(1-\frac{a}{z^2}\right)\left[(\dot{x}z)^2-z^2(\dot{x}^2+\dot{z}^2)\right]\right\}^{1/2}$$

$$a>0, \qquad K>0 \qquad (\text{"elastic constant"}) \tag{63}$$

subject to the two conditions

$$g^{\alpha\beta}\frac{\partial L}{\partial \dot{x}^\alpha}\frac{\partial L}{\partial \dot{x}^\beta}\in\mathbb{R}^+$$

$$\left(1-\frac{a}{z^2}\right)\left[(\dot{x}z)^2-z^2(\dot{x}^2+\dot{z}^2)\right]\in\mathbb{R}^+ \tag{64}$$

The second one of (64) is the regularity condition for L; the first one is easily recognized to be the mass condition, as we define

$$p_\alpha=-\frac{\partial L}{\partial \dot{x}^\alpha}, \qquad q_\alpha=-\frac{\partial L}{\partial \dot{z}^\alpha} \tag{65}$$

by which it becomes

$$p^2\gtrsim 0 \tag{66}$$

Calculating the Hessian matrix of L, we find that the rank is everywhere 6 in the (64) region. The two primary constraints can be expressed as

$$p^2+q^2+K^2z^2-m_0^2\approx 0 \qquad \left(m_0^2=K^2a\right)$$

$$pz\approx 0 \tag{67}$$

and they generate only the secondary constraint

$$pq\approx 0 \tag{68}$$

[7]The metric tensor, called $g^{\alpha\beta}$, will be diag$(+1,-1,-1,-1)$. Greek indices run through the values $0,1,2,3$ unless otherwise indicated.

The set of constraints is now complete, and already maximizes the number of first-class constraint functions: the only first-class constraint function is $\Omega = p^2 + q^2 + K^2z^2 - m_0^2$.

The Hamiltonian equations are

$$\dot{p}_\alpha \approx \frac{\partial}{\partial x^\alpha} H_T, \qquad \dot{q}_\alpha \approx \frac{\partial}{\partial z^\alpha} H_T$$
$$\dot{x}^\alpha \approx -\frac{\partial}{\partial p_\alpha} H_T, \qquad \dot{z}^\alpha \approx -\frac{\partial}{\partial q_\alpha} H_T \tag{69}$$
$$H_T = \lambda\left(p^2 + q^2 + K^2z^2 - m_0^2\right)$$

or, in explicit form,

$$\dot{p}_\alpha \approx 0, \qquad \dot{q}_\alpha \approx 2\lambda K^2 g_{\alpha\beta} z^\beta$$
$$\dot{x}^\alpha \approx -2\lambda g^{\alpha\beta} p_\beta, \qquad \dot{z}^\alpha \approx -2\lambda g^{\alpha\beta} p_\beta \tag{70}$$

and, using them, the second of (64) is reduced to

$$\lambda^2(p^2 + q^2) \underset{\sim}{\neq} 0 \tag{71}$$

Noting that $(p^2 + q^2) \underset{\sim}{\neq} 0$ is implicit in $pz \approx 0$, $p^2 \gtrsim 0$, $p^2 + q^2 + K^2z^2 - m_0^2 \approx 0$, the whole set of restrictions is rewritten as follows:

$$p^2 + q^2 + K^2z^2 - m_0^2 \approx 0, \qquad pz \approx 0$$
$$pq \approx 0, \qquad p^2 \gtrsim 0, \qquad \lambda \neq 0 \tag{72}$$

$p^2 \gtrsim 0$ is a constant of the motion, and for meeting the condition $\lambda \neq 0$ we simply need to use a $\bar{\lambda}(\tau)$: $\bar{\lambda}(\tau) \neq 0\ \forall\tau$.

The calculation of the total Lagrangian gives

$$L_T = (1/4\bar{\lambda})\left(\dot{x}^2 + \dot{z}^2 - 4K^2\bar{\lambda}^2z^2 + 4\bar{\lambda}^2m_0^2\right) \tag{73}$$

and this function is everywhere valid and regular, suitable for use in Feynman's propagator calculation.

Concerning quantization, with the usual definitions

$$\hat{a}^\mu = \left(\frac{K}{2}\right)^{1/2} \hat{z}^\mu - i g^{\mu\nu}\left(\frac{1}{2K}\right)^{1/2} \hat{q}_\nu \tag{74}$$

the problem including "observation" of both kinds of constraints is simply

$$i\frac{d}{d\tau}|\Phi(\tau)\rangle = 0, \qquad (\hat{p}^2 + \hat{q}^2 + K^2\hat{z}^2 - m_0^2)|\Phi(\tau)\rangle = 0$$

$$\hat{p}\hat{a}|\Phi(\tau)\rangle = 0, \qquad \langle\Phi(\tau)|\hat{p}^2|\Phi(\tau)\rangle > 0 \tag{75}$$

while the Schrödinger equation regulating generic representative states is

$$i\frac{d}{d\tau}|\Phi(\tau)\rangle = \bar{\lambda}(\tau)(\hat{p}^2 + \hat{q}^2 + K^2\hat{z}^2 - m_0^2)|\Phi(\tau)\rangle = 0 \tag{76}$$

If we also observe the momentum $\hat{p}$

$$\hat{p}_\mu|\Phi(\tau)\rangle = p_\mu|\Phi(\tau)\rangle \tag{77}$$

the equations in (75) can be solved in Kim and Noz's way (Kim and Noz, 1973). Introducing the usual operators of occupation:

$$\hat{N}^0 = \hat{a}^{0\dagger}\hat{a}^0, \qquad \hat{N}^k = \hat{a}^k\hat{a}^{k\dagger}, \qquad k = 1,2,3 \tag{78}$$

and passing from our frame of reference (say Σ) to a frame of reference Σ^{cm} in which $p_k = 0$ $(k = 1,2,3)$, the equations (75), (77) become simpler:

$$i\frac{d}{d\tau}|\Phi_{\mathrm{cm}}(\tau)\rangle = 0$$

$$(\hat{p}_0)^{\mathrm{cm}}|\Phi_{\mathrm{cm}}(\tau)\rangle = (p_0)^{\mathrm{cm}}|\Phi_{\mathrm{cm}}(\tau)\rangle \tag{79}$$

$$(\hat{p}_k)^{\mathrm{cm}}|\Phi_{\mathrm{cm}}(\tau)\rangle = 0, \qquad k = 1,2,3$$

$$\left\{2K\left[(\hat{N}^0)^{\mathrm{cm}} - \sum_{j=1}^{3}(\hat{N}^j)^{\mathrm{cm}} + (p_0^2)^{\mathrm{cm}} - m_0^2 - 2K\right]\right\}|\Phi_{\mathrm{cm}}(\tau)\rangle = 0$$

$$(p_0)^{\mathrm{cm}}(\hat{a}^0)^{\mathrm{cm}}|\Phi_{\mathrm{cm}}(\tau)\rangle = 0 \Rightarrow (\hat{N}^0)^{\mathrm{cm}}|\Phi_{\mathrm{cm}}(\tau)\rangle = 0$$

and the normalized wave function can be solved in the following form:

$$\Phi^{\mathrm{cm}}(x^{\mathrm{cm}}, z^{\mathrm{cm}}) = \frac{K}{4\pi^3(2^n n'!n''!n'''!)^{1/2}}\exp\left[-i(p_0)^{\mathrm{cm}}(x^0)^{\mathrm{cm}}\right]$$
$$\times\exp\left[-\tfrac{1}{2}K\delta_{\mu\nu}(z^\mu)^{\mathrm{cm}}(z^\nu)^{\mathrm{cm}}\right]H_{n'}\left(K^{1/2}(-z_1)^{\mathrm{cm}}\right)$$
$$\times H_{n''}\left(K^{1/2}(-z_2)^{\mathrm{cm}}\right)H_{n'''}\left(K^{1/2}(-z_3)^{\mathrm{cm}}\right) \tag{80}$$

in which

$$\left(p_0^2\right)^{\mathrm{cm}}=m_0^2+2K+2Kn$$

where $n=(n'+n''+n''')$, and n', n'', n''' are the eigenvalues corresponding to the observations

$$\begin{gathered}(\hat{N}^j)^{\mathrm{cm}}|\Phi_{\mathrm{cm}}\rangle=n^{(j)}|\Phi_{\mathrm{cm}}\rangle, \qquad j=1,2,3\\ n'=n^{(1)}, \qquad n''=n^{(2)}, \qquad n'''=n^{(3)}\end{gathered} \tag{81}$$

Deriving the wave function in Σ [with a scalar transformation from (80)] and using a symbol $\Phi_{p\mathbf{n}}$ to point out the quantum numbers p and $\mathbf{n}\equiv(n', n'', n''')$, we have

$$\begin{aligned}\Phi_{p\mathbf{n}}(x,z)=&\frac{K}{4\pi^3(2^n n'!n''!n'''!)^{1/2}}\exp(-\mathrm{i}px)\\ &\times\exp\left\{\frac{1}{2}K\left[z^2-\frac{2(pz)^2}{p^2}\right]\right\}\\ &\times H_{n'}\left(K^{1/2}\lambda^1{}_\nu z^\nu\right)H_{n''}\left(K^{1/2}\lambda^2{}_\nu z^\nu\right)H_{n'''}\left(K^{1/2}\lambda^3{}_\nu z^\nu\right)\end{aligned} \tag{82}$$

where

$$p^2=b^2+2Kn \qquad \left(b^2=m_0^2+2K\right)$$

and where the $\lambda(p)$ coefficients are defined by the Lorentz transformation

$$(x^\mu)^{\mathrm{cm}}=\lambda^\mu{}_\nu(p)x^\nu \tag{83}$$

If we now define the classical functions

$$\begin{gathered}S_k=\tfrac{1}{2}\varepsilon_{kij}S^{ij} \qquad (i,j,k=1,2,3)\\ \mathbf{S}^2=\sum_{k=1}^{3}(S_k)^2\end{gathered} \tag{84}$$

where ε_{kij} is the total antisymmetric three-dimensional symbol and where

$$S^{\mu\nu}=-g^{\mu\alpha}z^\nu q_\alpha+g^{\nu\alpha}z^\mu q_\alpha \tag{85}$$

then the S_k's and $\mathbf{S}^2$ are classically constants of the motion. The corresponding quantum operators, which can be expressed using the forms

$$\hat{S}_k = \frac{i}{2}\varepsilon_{kij}(\hat{a}^j\hat{a}^{i\dagger} - \hat{a}^i\hat{a}^{j\dagger}) \tag{86}$$

and satisfy ($\hat{0}$ is the null operator)

$$\begin{aligned} \left[\hat{S}_i, \hat{S}_j\right] &= i\varepsilon_{ijk}\hat{S}_k \\ \left[\hat{S}_i, \hat{\mathbf{S}}^2\right] &= \hat{0} \end{aligned} \tag{87}$$

can be considered operators of integer spin, and the two observations

$$\begin{aligned} (\hat{\mathbf{S}}^2)^{\mathrm{cm}}|\Phi_{\mathrm{cm}}\rangle &= s(s+1)|\Phi_{\mathrm{cm}}\rangle, \qquad s\in\mathbb{N}+\{0\} \\ (\hat{S}_3)^{\mathrm{cm}}|\Phi_{\mathrm{cm}}\rangle &= s_3|\Phi_{\mathrm{cm}}\rangle, \qquad -s \leqslant s_3 \leqslant s, \qquad s_3\in\mathbb{Z}+\{0\} \end{aligned} \tag{88}$$

are possible and compatible with equations (75) and (77), to replace (81). This means, using Kim and Noz's states, that linear combinations of them can provide states of definite spin. For instance,

$$\begin{aligned} |\Phi_{pss_3}\rangle_{\substack{s=0\\ s_3=0}} &= |\Phi_{p\mathbf{n}}\rangle_{\mathbf{n}\equiv(0,0,0)} \\ |\Phi_{pss_3}\rangle_{\substack{s=1\\ s_3=1}} &= \frac{1}{\sqrt{2}}|\Phi_{p\mathbf{n}}\rangle_{\mathbf{n}\equiv(1,0,0)} + \frac{i}{\sqrt{2}}|\Phi_{p\tilde{\mathbf{n}}}\rangle_{\tilde{\mathbf{n}}\equiv(0,1,0)} \\ |\Phi_{pss_3}\rangle_{\substack{s=1\\ s_3=0}} &= |\Phi_{p\mathbf{n}}\rangle_{\mathbf{n}\equiv(0,0,1)} \\ |\Phi_{pss_3}\rangle_{\substack{s=1\\ s_3=-1}} &= \frac{1}{\sqrt{2}}|\Phi_{p\mathbf{n}}\rangle_{\mathbf{n}\equiv(1,0,0)} - \frac{i}{\sqrt{2}}|\Phi_{p\tilde{\mathbf{n}}}\rangle_{\tilde{\mathbf{n}}=(0,1,0)} \end{aligned} \tag{89}$$

with obvious meaning of the symbols employed.

That concludes our resumé of the model. Next we use principle (55) for the Feynman propagator calculation.

Since L_T is a quadratic function [see formula (73)] the expression of the Feynman propagator

$$\begin{aligned} \langle x''z''|\hat{U}_{\tau''}(\tau')|x'z'\rangle = \int_{\substack{x(\tau')=x',\, z(\tau')=z'\\ x(\tau'')=x'',\, z(\tau'')=z''}} &\exp\left[i\int_{\tau'}^{\tau''} L_T(\dot{x}, \dot{z}, x, z, \bar{\lambda})\, d\tau\right] \\ &\times \mathfrak{D}(x(\tau)z(\tau)) \end{aligned} \tag{90}$$

can be simply calculated in the following way (Feynman and Hibbs, 1965):

$$\langle x''z''|\hat{U}_{\tau''}(\tau')|x'z'\rangle = f(\tau', \tau'')\exp\left(i\int_{\tau'}^{\tau''} L_T\, d\tau\right)\Bigg|_{\substack{x(\tau')=x',\, z(\tau')=z' \\ x(\tau'')=x'',\, z(\tau'')=z''}} \tag{91}$$

where the integral has to be performed placing in L_T the solution $(x(\tau), z(\tau))$ of classical problem (70) solved with the conditions indicated in the above formula, and where $f(\tau', \tau'')$ is a coefficient so that the propagator satisfies

$$\int \langle x''z''|\hat{U}_{\tau''}(\tau')|xz\rangle\langle x'z'|\hat{U}_{\tau''}(\tau')|xz\rangle^* \, dx\, dz = \delta(x''-x')\,\delta(z''-z')$$

$$\lim_{\tau''\to\tau'} \langle x''z''|\hat{U}_{\tau''}(\tau')|x'z'\rangle = \delta(x''-x')\delta(z''-z') \tag{92}$$

$$\int \langle x''z''|\hat{U}_{\tau''}(\tau_0)|xz\rangle\langle xz|\hat{U}_{\tau_0}(\tau')|x'z'\rangle \, dx\, dz = \langle x''z''|\hat{U}_{\tau''}(\tau')|x'z'\rangle \qquad \forall\tau_0$$

Choosing $\lambda = -1/2m_0$ [that means, in (70), $\dot{x}^\alpha \approx p^\alpha/m_0$, i.e., τ is proportional by a constant factor to the proper time measured in x] and forgetting an extra multiplicative factor (not depending on x', x'', z', z'') the integral in (91) is

$$\begin{aligned}
&\exp\left(i\int_{\tau'}^{\tau''} L_T\, d\tau\right)\Bigg|_{\substack{x(\tau')=x',\, z(\tau')=z' \\ x(\tau'')=x'',\, z(\tau'')=z''}} \\
&\quad\propto \exp\left\{-\frac{im_0}{2}(\tau''-\tau')\left[\left(\frac{x''-x'}{\tau''-\tau'}\right)^2+1\right]\right\} \\
&\quad\times\exp\left\{-\frac{iK}{2}\left[\left(\operatorname{cotan}\frac{K(\tau''-\tau')}{m_0}\right)(z''^2+z'^2)\right.\right. \\
&\qquad\left.\left. -\frac{2}{\sin[K(\tau''-\tau')/m_0]}z''z'\right]\right\}
\end{aligned} \tag{93}$$

With other calculations, using (92), the complete Feynman propagator is expressed as

$$\langle x''z''|\hat{U}_{\tau''}(\tau')|x'z'\rangle = [\text{Right-hand side of (93)}]$$

$$\times\,\varphi(\tau',\tau'')\frac{-\mathrm{i}m_0^2}{4\pi^2(\tau''-\tau')|\tau''-\tau'|}$$

$$\times\,\frac{-\mathrm{i}K^2}{4\pi^2\left(\sin\dfrac{K(\tau''-\tau')}{m_0}\right)\left|\sin\dfrac{K(\tau''-\tau')}{m_0}\right|} \tag{94}$$

where $\varphi(\tau',\tau'')$ is an arbitrary phase factor, not determinable by (92) and of no physical meaning, which we can choose equal to 1.

Now the real thing needed for an explicit scattering treatment of particles with covariant harmonic oscillator model, is the vertex function, and we will give here its expression, referring to the Appendix for details of calculations.

Using the same conventions applied in describing Figure 1, the vertex function

$$\langle \rho_{(0)}|\Phi_{(1)}\Phi_{(2)}\rangle|_{\substack{\text{interaction}\\ \text{at } P}} \tag{95}$$

corresponds to a graph of three covariant harmonic oscillator systems, as shown in Figure 2, where we have distinguished each system into its two subcomponents. The states $|\Phi_{(i)}\rangle$ of external lines $[i]$ $(i=1,2)$ are to be physical states: if $p_{(i)}$ and $\mathbf{n}_{(i)}\equiv(n'_{(i)}, n''_{(i)}, n'''_{(i)})$ are their quantum numbers $(p^2_{(i)}=b^2_{(i)}+2K_{(i)}n_{(i)})$, those states will be called, as usual, $|\Phi_{p_{(i)}\mathbf{n}_{(i)}}\rangle$.

The ket $|\rho_{(0)}\rangle$ has to belong to a set able to span the Hilbertian space of virtual kets of the [0] system. We will choose, for $|\rho_{(0)}\rangle$, a form indicated as $|p_{(0)}\mathbf{n}_{(0)}\rangle$, corresponding to a Kim and Noz state with quantum numbers $p_{(0)}, \mathbf{n}_{(0)}$, deprived of the mass shell condition $p^2_{(0)}=b^2_{(0)}+2K_{(0)}n_{(0)}$. Our choice, since $p^2_{(0)}>0$ has to be met, means we exclude, in the propagation, photon or tachyonlike virtual particles. And since we have

$$\left(\hat{N}^0_{(0)}\right)^{\mathrm{cm}_{(0)}}|(p_{(0)}\mathbf{n}_{(0)})_{\mathrm{cm}_{(0)}}\rangle = 0 \tag{96}$$

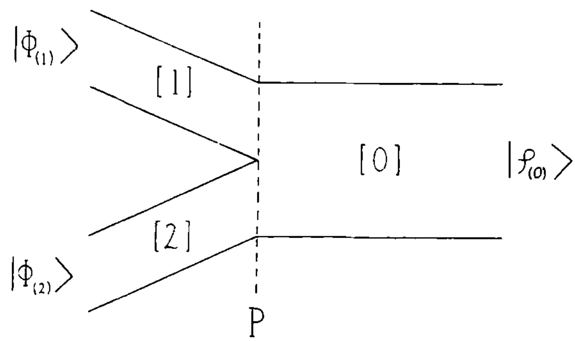

Fig. 2. Vertex of three oscillator systems.

we exclude $(z^0_{(0)})^{\mathrm{cm}_{(0)}}$ excited states too, which corresponds (Kalb and Van Alstine, 1976) to excluding the propagation of particles that, in shell, might be photon or tachyonlike. The second assumption may be considered forced by the physical meaning, while the first one clearly restricts us to study cases of scattering for which

$$(p_{(1)}+p_{(2)})^2>0 \tag{97}$$

Having established the above and going back to vertex (95) we have

$$\langle p_{(0)}\mathbf{n}_{(0)}|\Phi_{p_{(1)}\mathbf{n}_{(1)}}\Phi_{p_{(2)}\mathbf{n}_{(2)}}\rangle\Big|_{\substack{\text{interaction}\\ \text{at } P}}=\frac{4C'}{9\pi K}\delta(p_{(1)}+p_{(2)}-p_{(0)})$$

$$\times\exp\left\{\frac{(p^4_{(1)}+p^4_{(2)}+p^4_{(0)})^2+p^2_{(1)}p^2_{(2)}p^2_{(0)}(p^2_{(1)}+p^2_{(2)}+p^2_{(0)})}{8K\big[(p^2_{(0)}+p^2_{(2)}-p^2_{(1)})(p^2_{(0)}-p^2_{(2)}+p^2_{(1)})(-p^2_{(0)}+p^2_{(2)}+p^2_{(1)})-p^2_{(1)}p^2_{(2)}p^2_{(0)}\big]}\right\}$$

$$\times\exp\left\{\frac{-2p^2_{(1)}p^2_{(2)}(p^4_{(1)}+p^4_{(2)})-2p^2_{(1)}p^2_{(0)}(p^4_{(1)}+p^4_{(0)})-2p^2_{(2)}p^2_{(0)}(p^4_{(2)}+p^4_{(0)})}{8K\big[(p^2_{(0)}+p^2_{(2)}-p^2_{(1)})(p^2_{(0)}-p^2_{(2)}+p^2_{(1)})(-p^2_{(0)}+p^2_{(2)}+p^2_{(1)})-p^2_{(1)}p^2_{(2)}p^2_{(0)}\big]}\right\}$$

$$\times\left[\left(\prod_{\sigma=0}^{2}\frac{1}{(2^{n_{(\sigma)}}n'_{(\sigma)}!n''_{(\sigma)}!n'''_{(\sigma)}!)^{1/2}}\frac{\partial^{n_{(\sigma)}}}{\partial J'^{n'_{(\sigma)}}_{(\sigma)}\partial J''^{n''_{(\sigma)}}_{(\sigma)}\partial J'''^{n'''_{(\sigma)}}_{(\sigma)}}\right)\right.$$

$$\times\exp\left\{\tilde{J}_{(1)\mu}(-KX^{\mu\nu}+g^{\mu\nu})\tilde{J}_{(1)\nu}+\tilde{J}_{(2)\mu}\left(-K\overset{1\leftrightarrow 2}{X^{\mu\nu}}+g^{\mu\nu}\right)\tilde{J}_{(2)\nu}\right.$$

$$+\tilde{J}_{(0)\mu}\left[-K\left(X^{\mu\nu}+\overset{1\leftrightarrow 2}{X^{\mu\nu}}+2Z^{\mu\nu}\right)+g^{\mu\nu}\right]\tilde{J}_{(0)\nu}+\tilde{J}_{(1)\mu}(-2KZ^{\mu\nu})\tilde{J}_{(2)\nu}$$

$$+\tilde{J}_{(1)\mu}[-2K(X^{\mu\nu}+Z^{\mu\nu})]\tilde{J}_{(0)\nu}+\tilde{J}_{(2)\mu}\left[-2K\left(\overset{1\leftrightarrow 2}{X^{\mu\nu}}+Z^{\mu\nu}\right)\right]\tilde{J}_{(0)\nu}$$

$$+p_{(1)\mu}\left(-\frac{\mathrm{i}K^{1/2}}{2}Z^{\mu\nu}\right)\tilde{J}_{(1)\nu}+p_{(2)\mu}\left(\frac{\mathrm{i}K^{1/2}}{2}X^{\mu\nu}\right)\tilde{J}_{(1)\nu}$$

$$+p_{(1)\mu}\left(-\frac{\mathrm{i}K^{1/2}}{2}\overset{1\leftrightarrow 2}{X^{\mu\nu}}\right)\tilde{J}_{(2)\nu}+p_{(2)\mu}\left(\frac{\mathrm{i}K^{1/2}}{2}Z^{\mu\nu}\right)\tilde{J}_{(2)\nu}$$

$$+p_{(1)\mu}\left[-\frac{\mathrm{i}K^{1/2}}{2}\left(\overset{1\leftrightarrow 2}{X^{\mu\nu}}+Z^{\mu\nu}\right)\right]\tilde{J}_{(0)\nu}$$

$$+p_{(2)\mu}\left[\frac{\mathrm{i}K^{1/2}}{2}(X^{\mu\nu}+Z^{\mu\nu})\right]\tilde{J}_{(0)\nu}\Bigg\}\Bigg]\Bigg|_{\substack{J_{(\sigma)}=0,\\ \sigma=0,1,2}} \tag{98}$$

where

(i) C' is a coupling constant.
(ii) K is the elastic constant of the three systems, supposed to be the same for all of them ($K_{(\sigma)}=K$, $\sigma=0,1,2$).
(iii) the quantum conditions $p_{(i)}^2=b_{(i)}^2+2Kn_{(i)}$: $n_{(i)}=(n'_{(i)}+n''_{(i)}+n'''_{(i)})$, $i=1,2$, are to be remembered.
(iv) $J_{(\sigma)}$ is a four-component object $(J_{(\sigma)\mu})=(0, J'_{(\sigma)}, J''_{(\sigma)}, J'''_{(\sigma)})$: $J_{(\sigma)}^2=-J'^2_{(\sigma)}-J''^2_{(\sigma)}-J'''^2_{(\sigma)}$.
(v) $\tilde{J}_{(\sigma)\mu}=J_{(\sigma)\nu}\Lambda^\nu_\mu(p_{(\sigma)})$.
(vi) $(X^{\mu\nu}),(Z^{\mu\nu})$ are two 4×4 matrices, functions of $p_{(1)}, p_{(2)}, p_{(0)}$ (see the Appendix for their explicit expressions).
(vii) $(\overset{1\leftrightarrow 2}{X^{\mu\nu}})$, is the same matrix as above, with $p_{(1)}$ and $p_{(2)}$ interchanged.
(viii) The expression

$$\frac{\partial^{n_{(\sigma)}}}{\partial J'^{n'_{(\sigma)}}_{(\sigma)}\partial J''^{n''_{(\sigma)}}_{(\sigma)}\partial J'''^{n'''_{(\sigma)}}_{(\sigma)}} \tag{99}$$

refers to the $\mathbf{n}_{(\sigma)}$ state by a partial derivative of the $n_{(\sigma)}$ order, performed $n'_{(\sigma)}$ times on $J'_{(\sigma)}$, $n''_{(\sigma)}$ times on $J''_{(\sigma)}$, and $n'''_{(\sigma)}$ times on $J'''_{(\sigma)}$. After all the derivatives are calculated the variables $J_{(\sigma)}$ disappear in virtue of the $J_{(\sigma)}=0$ conditions.

It is to be noticed that expression (98), very complicated indeed, is, on the other hand, able to generate vertex functions for interaction of whatever spin numbers we may consider for the three systems. In the next section I will examine the vertex of three scalars and a vertex of two scalars and a spin-1 boson.

5. EXPLICIT FORMULATION OF THE VERTEX IN TWO PHYSICAL CASES, AND CONCLUSIONS

If we use the generating formula (98) and take $\mathbf{n}_{(\sigma)} \equiv (0,0,0)$, $\sigma = 0,1,2$, according to expressions (89) we get a vertex of three scalars. [(89) apply to off-shell states too, of the kind of $|p_{(0)}\mathbf{n}_{(0)}\rangle$ involved here.] The explicit formulation of that vertex is

$$\mathcal{V}(p_{(0)}; s_{(0)}=0; p_{(1)}; s_{(1)}=0; p_{(2)}; s_{(2)}=0)$$

$$= \frac{\gamma'}{(2\pi)^2}\delta(p_{(1)}+p_{(2)}-p_{(0)})E(p_{(1)}, p_{(2)}, p_{(0)}) \tag{100}$$

where we define the new coupling constant

$$\gamma' = (2\pi)^2 \frac{4C'}{9\pi K} \tag{101}$$

and where E is the product of the first two exponential factors in (98) (calculated, in this case, with $p_{(i)}^2 = b_{(i)}^2$, $i = 1,2$).

In an analogous way, considering the three cases: $\mathbf{n}_{(1)} \equiv (1,0,0)$, $\mathbf{n}_{(1)} \equiv (0,1,0)$, $\mathbf{n}_{(1)} \equiv (0,0,1)$, with $\mathbf{n}_{(0)} = \mathbf{n}_{(2)} \equiv (0,0,0)$, we get

$$\mathcal{V}(p_{(0)}; s_{(0)}=0; p_{(1)}; s_{(1)}=1; s_{3(1)}=u; p_{(2)}; s_{(2)}=0)$$

$$= -\frac{\gamma'}{(2\pi)^2}\delta(p_{(1)}+p_{(2)}-p_{(0)})E(p_{(1)}, p_{(2)}, p_{(0)})P(p_{(1)}, p_{(2)}, p_{(0)})$$

$$\times \frac{i\sqrt{2}}{K^{1/2}} g^{\alpha\nu} p_{(2)\alpha} F_\nu(p_{(1)}, u) \tag{102}$$

where

$$p_{(1)}^2 = b_{(1)}^2 + 2K, \qquad p_{(2)}^2 = b_{(2)}^2 \tag{103}$$

$$F_\nu(p_{(1)}, u) = \begin{cases} \frac{1}{\sqrt{2}}\lambda_\nu^1(p_{(1)}) + \frac{i}{\sqrt{2}}\lambda_\nu^2(p_{(1)}) & \text{if } u = +1 \\ \lambda_\nu^3(p_{(1)}) & \text{if } u = 0 \\ \frac{1}{\sqrt{2}}\lambda_\nu^1(p_{(1)}) - \frac{i}{\sqrt{2}}\lambda_\nu^2(p_{(1)}) & \text{if } u = -1 \end{cases} \tag{104}$$

$$P(p_{(1)}, p_{(2)}, p_{(0)}) = \frac{p_{(1)}^2(p_{(1)}^2 - p_{(2)}^2)(p_{(1)}^2 - p_{(0)}^2)}{\left[(p_{(0)}^2 + p_{(1)}^2 - p_{(2)}^2)(p_{(0)}^2 - p_{(1)}^2 + p_{(2)}^2)(-p_{(0)}^2 + p_{(1)}^2 + p_{(2)}^2) - p_{(1)}^2 p_{(2)}^2 p_{(0)}^2\right]} \tag{105}$$

Observing that

$$g^{\alpha\nu}\lambda_\nu^k(p_{(1)})p_{(1)\alpha} = 0, \qquad k = 1,2,3 \tag{106}$$

we can write (using also the δ function condition $p_{(0)} = p_{(1)} + p_{(2)}$)

$$g^{\alpha\nu}p_{(2)\alpha}F_\nu(p_{(1)}, u) = \tfrac{1}{2}g^{\alpha\nu}(p_{(2)\alpha} + p_{(0)\alpha})F_\nu(p_{(1)}, u) \tag{107}$$

Defining

$$\varepsilon_u^\alpha(p_{(1)}) = g^{\alpha\nu}F_\nu(p_{(1)}, u), \qquad u = -1, 0, +1 \tag{108}$$

[which meets the properties $\varepsilon_u^\alpha(p_{(1)})p_{(1)\alpha} = 0$ and $g_{\alpha\beta}\varepsilon_u^\alpha(p_{(1)})\varepsilon_{u'}^{\beta *}(p_{(1)}) = -\delta_{uu'}$] we have

$$\begin{aligned} &\mathcal{V}(p_{(0)}; s_{(0)} = 0; p_{(1)}; s_{(1)} = 1; s_{3(1)} = u; p_{(2)}; s_{(2)} = 0) \\ &\quad = -\frac{i\sqrt{2}}{2K^{1/2}}\frac{\gamma'}{(2\pi)^2}\delta(p_{(1)} + p_{(2)} - p_{(0)}) \\ &\qquad \times E(p_{(1)}, p_{(2)}, p_{(0)})P(p_{(1)}, p_{(2)}, p_{(0)})(p_{(2)} + p_{(0)})_\alpha \varepsilon_u^\alpha(p_{(1)}) \end{aligned} \tag{109}$$

It is easy to interchange the roles of particles [1] and [2], since this only requires

$$p_{(1)} \leftrightarrow p_{(2)}, \qquad \mathrm{i} \to -\mathrm{i} \tag{110}$$

Now expressions (100) and (109) are similar to those of usual field theories, apart from the extra factors E in the first case and $E \cdot P$ in the second case.

The E factor satisfies

$$\lim_{K \to +\infty} E = 1 \tag{111}$$

i.e., in the limit of rigid bodies, vertex (100) reduces exactly to the usual one.

Furthermore, provided $p_{(1)}^2 > 0$, $p_{(2)}^2 > 0$, we have

$$\lim_{p_{(1)} \to \infty} E(p_{(1)}, p_{(2)}, p_{(1)} + p_{(2)}) = \lim_{p_{(2)} \to \infty} E(p_{(1)}, p_{(2)}, p_{(1)} + p_{(2)}) = 0 \tag{112}$$

and both limits go to zero like

$$\lim_{x \to \infty} \exp\{-x^2\}, \qquad x \in \mathbb{R} \tag{113}$$

That means the E factor is a strong convergence factor for large timelike momenta. As for graphs of higher orders than the one shown in Figure 1, there is the possibility they need no renormalization. All the theory here developed is rather far from a physical description of real particles, but what is shown is attractive. In fact it seems there is the possibility of having finite theories, using composite models of particles. What is needed is a composite model that is more general and more physical than the harmonic oscillator one, and, if possible, which leads to fewer complications in the algebra.

APPENDIX

Remembering the generating formula for Hermite polynomials

$$H_l(t) = \left[\frac{\partial^l}{\partial j^l} \exp(-j^2 + 2jt)\right]\Bigg|_{j=0}, \qquad l \in \mathbb{N} + \{0\}, \qquad t \in \mathbb{C} \tag{A.1}$$

the wave function $\langle xz|p\mathbf{n}\rangle$ can be expressed in the following way:

$$\langle xz|p\mathbf{n}\rangle = \frac{K}{4\pi^3(2^n n'!n''!n'''!)^{1/2}}\exp(-\mathrm{i}px)\exp\left\{\frac{1}{2}K\left[z^2-\frac{2(pz)^2}{p^2}\right]\right\}$$
$$\times\left\{\frac{\partial^n}{\partial J'^{n'}\partial J''^{n''}\partial J'''^{n'''}}\exp\left[J^2+2K^{1/2}J_\alpha\lambda_\nu^\alpha(p)z^\nu\right]\right\}\Bigg|_{J=0} \qquad \text{(A.2)}$$

where we have introduced a four-dimensional vector J whose covariant space components are J', J'', J''', and whose square J^2 is to be calculated in the pseudo-Euclidean way, and the covariant time component may be chosen arbitrarily (for instance zero, as indicated in Section 4). When we want to apply the vertex function expression (57) to the case of a vertex of three oscillator systems, we make use of (A.2) and so the first expression we get is

$$\langle p_{(0)}\mathbf{n}_{(0)}|\Phi_{p_{(1)}\mathbf{n}_{(1)}}\Phi_{p_{(2)}\mathbf{n}_{(2)}}\rangle\Big|_{\substack{\text{interaction}\\ \text{at } P}} = C\frac{K^3}{(4\pi^3)^3\left[\prod_{\sigma=0}^{2}\left(2^{n_{(\sigma)}}n'_{(\sigma)}!n''_{(\sigma)}!n'''_{(\sigma)}!\right)^{1/2}\right]}$$
$$\times\left(\prod_{\sigma=0}^{2}\frac{\partial^{n_{(\sigma)}}}{\partial J'^{n'_{(\sigma)}}_{(\sigma)}\partial J''^{n''_{(\sigma)}}_{(\sigma)}\partial J'''^{n'''_{(\sigma)}}_{(\sigma)}}\right)\Bigg|_{\substack{J_{(\sigma)}=0,\\ \sigma=0,1,2}}\exp\left(\sum_{\sigma=0}^{2}J^2_{(\sigma)}\right)$$
$$\times\int\left\{\exp\left(\sum_{\sigma=0}^{2}\left[\frac{K}{2}z^2_{(\sigma)}-\frac{K}{p^2_{(\sigma)}}(p_{(\sigma)}z_{(\sigma)})^2+2K^{1/2}\tilde{J}_{(\sigma)}z_{(\sigma)}\right]\right)\right.$$
$$\left.\times\exp\left(\mathrm{i}p_{(0)}x_{(0)}-\mathrm{i}p_{(1)}x_{(1)}-\mathrm{i}p_{(2)}x_{(2)}\right)\right\}\Bigg|_{\substack{\text{with connections}\\ \text{in } P}} d\begin{pmatrix}\text{variables}\\ \text{of connection}\end{pmatrix}$$

(A.3)

where

$$p^2_{(i)} = b^2_{(i)} + 2Kn_{(i)}, \qquad i=1,2$$

and, for any problem of interpretation, we refer to the list of definitions in the last part of Section 4.

Considering Figure 2 and calling $\tilde{x}_{(\sigma)1}$ and $\tilde{x}_{(\sigma)2}$ the four-dimensional coordinates of the two subcomponents of the oscillator system $[\sigma]$, $\sigma=0,1,2$, the connections in P can be written in the following way:

$$\begin{aligned}\tilde{x}_{(1)1} &= \tilde{x}_{(0)1} = y' \\ \tilde{x}_{(1)2} &= \tilde{x}_{(2)1} = y'' \\ \tilde{x}_{(2)2} &= \tilde{x}_{(0)2} = y'''\end{aligned} \tag{A.4}$$

that is, using variables $x_{(\sigma)}$, $z_{(\sigma)}$ [remember $x_{(\sigma)} = \frac{1}{2}(\tilde{x}_{(\sigma)1} + \tilde{x}_{(\sigma)2})$; $z_{(\sigma)} = \tilde{x}_{(\sigma)2} - \tilde{x}_{(\sigma)1}$]

$$\begin{aligned} x_{(0)} &= \frac{y' + y'''}{2}, \qquad z_{(0)} = y''' - y' \\ x_{(1)} &= \frac{y' + y''}{2}, \qquad z_{(1)} = y'' - y' \\ x_{(2)} &= \frac{y'' + y'''}{2}, \qquad z_{(2)} = y''' - y'' \end{aligned} \tag{A.5}$$

Introducing the above connections in formula (A.3), then changing the variables of integration y', y'', y''' to the new ones

$$\begin{aligned} x &= \frac{y' + y'''}{2} \\ y_1 &= y'' - y' \\ y_2 &= y''' - y'' \end{aligned} \tag{A.6}$$

including the Jacobian of this transformation in the coupling constant (which we will now call C'), and performing a first integration in dx, we get

$$\left\langle p_{(0)}\mathbf{n}_{(0)} \middle| \Phi_{p_{(1)}\mathbf{n}_{(1)}} \Phi_{p_{(2)}\mathbf{n}_{(2)}} \right\rangle \Big|_{\substack{\text{interaction} \\ \text{at } P}} = C' \frac{K^3 (2\pi)^4}{(4\pi^3)^3 \prod_{\sigma=0}^{2} \left(2^{n_{(\sigma)}} n'_{(\sigma)}! n''_{(\sigma)}! n'''_{(\sigma)}!\right)^{1/2}}$$

$$\times \left(\prod_{\sigma=0}^{2} \frac{\partial^{n_{(\sigma)}}}{\partial J'^{n'_{(\sigma)}} \partial J''^{n''_{(\sigma)}} \partial J'''^{n'''_{(\sigma)}}} \right) \Bigg|_{\substack{J_{(\sigma)} = 0, \\ \sigma = 0,1,2}} \exp\left(\sum_{\sigma=0}^{2} J_{(\sigma)}^2 \right) \delta(p_{(1)} + p_{(2)} - p_{(0)})$$

$$\times \int \exp\Big[K y_1^2 + K y_2^2 + K y_1 y_2 + \left(2\tilde{J}_{(1)} K^{1/2} + 2\tilde{J}_{(0)} K^{1/2} - \frac{i}{2} p_{(2)} \right) y_1$$

$$+ \left(2\tilde{J}_{(2)} K^{1/2} + 2\tilde{J}_{(0)} K^{1/2} + \frac{i}{2} p_{(1)} \right) y_2 - \frac{K}{p_{(1)}^2} (p_{(1)} y_1)^2$$

$$-\frac{K}{p_{(2)}^2}(p_{(2)}y_2)^2-\frac{K}{p_{(0)}^2}(p_{(0)}y_1)^2$$

$$\left.-\frac{K}{p_{(0)}^2}(p_{(0)}y_2)^2-\frac{2K}{p_{(0)}^2}(p_{(0)}y_1)(p_{(0)}y_2)\right]dy_1\,dy_2 \tag{A.7}$$

where

$$p_{(i)}^2=b_{(i)}^2+2Kn_{(i)}, \qquad i=1,2$$

We define now the following eight-dimensional formalism:

$$\mathrm{Y}=(\mathrm{Y}^l)=\begin{pmatrix}(y_1^{\mu})\\ \hline (y_2^{\mu})\end{pmatrix} \qquad (l=1\cdots 8)$$

$$\mathrm{V}=(\mathrm{V}_l)=\begin{pmatrix}\left(2\tilde{J}_{(1)\mu}K^{1/2}+2\tilde{J}_{(0)\mu}K^{1/2}-\frac{1}{2}\mathrm{i}\,p_{(2)\mu}\right)\\ \hline \left(2\tilde{J}_{(2)\mu}K^{1/2}+2\tilde{J}_{(0)\mu}K^{1/2}+\frac{1}{2}\mathrm{i}\,p_{(1)\mu}\right)\end{pmatrix} \tag{A.8}$$

$$\mathrm{A}=(\mathrm{A}_{lm})=\left(\begin{array}{c|c}A_1 & \bar{C}\\ \hline \bar{C} & A_2\end{array}\right) \qquad (l,m=1\cdots 8)$$

where

$$\bar{C}=\left(\bar{C}_{\mu\nu}\right):\ \bar{C}_{\mu\nu}=\frac{K}{2}g_{\mu\nu}-\frac{K}{p_{(0)}^2}p_{(0)\mu}p_{(0)\nu}$$

$$A_1=\left(A_{1\mu\nu}\right):\ A_{1\mu\nu}=Kg_{\mu\nu}-\frac{K}{p_{(1)}^2}p_{(1)\mu}p_{(1)\nu}-\frac{K}{p_{(0)}^2}p_{(0)\mu}p_{(0)\nu}$$

$$A_2=\left(A_{2\mu\nu}\right):\ A_{2\mu\nu}=Kg_{\mu\nu}-\frac{K}{p_{(2)}^2}p_{(2)\mu}p_{(2)\nu}-\frac{K}{p_{(0)}^2}p_{(0)\mu}p_{(0)\nu}$$

and so the integral to be calculated in (A.7) is reduced to the form

$$\int\exp\left(\mathrm{A}_{lm}\mathrm{Y}^l\mathrm{Y}^m+\mathrm{V}_l\mathrm{Y}^l\right)d\mathrm{Y} \qquad (l,m=1\cdots 8) \tag{A.9}$$

Using also the δ condition (that is remembering, when useful, $p_{(0)}=p_{(1)}+$

$p_{(2)}$) we can find

$$\det(\mathrm{A}) = \left(\frac{3K^2}{4}\right)^4$$

$$\mathrm{A}^{-1} = \left(\begin{array}{c|c} X & Z \\ \hline T & R \end{array}\right) \tag{A.10}$$

where

$$X = (X^{\mu\nu}):\ X^{\mu\nu} = \frac{4}{3K} g^{\mu\nu} + x^{ij} g^{\mu\alpha} g^{\nu\beta} p_{(i)\alpha} p_{(j)\beta}$$

$$x^{11} = -\frac{4}{3K\Delta}\left[p_{(2)}^2\left(p_{(1)}^2 + p_{(2)}^2\right) - 2p_{(2)}^2 p_{(0)}^2 - 3\left(p_{(1)}^2 - p_{(0)}^2\right)^2\right]$$

$$x^{12} = x^{21} = -\frac{4}{3K\Delta}\left[\left(p_{(0)}^2 + p_{(2)}^2 - p_{(1)}^2\right)\left(2p_{(2)}^2 - 2p_{(0)}^2 + p_{(1)}^2\right) - p_{(1)}^2 p_{(2)}^2\right]$$

$$x^{22} = -\frac{4}{3K\Delta}\left[p_{(1)}^2\left(p_{(0)}^2 + p_{(2)}^2\right) - 2p_{(1)}^4\right] \tag{A.11}$$

$$\Delta = \left(p_{(0)}^2 + p_{(2)}^2 - p_{(1)}^2\right)\left(p_{(0)}^2 - p_{(2)}^2 + p_{(1)}^2\right)\left(-p_{(0)}^2 + p_{(2)}^2 + p_{(1)}^2\right) - p_{(1)}^2 p_{(2)}^2 p_{(0)}^2$$

$$Z = (Z^{\mu\nu}):\ Z^{\mu\nu} = -\frac{2}{3K} g^{\mu\nu} + z^{ij} g^{\mu\alpha} g^{\nu\beta} p_{(i)\alpha} p_{(j)\beta}$$

$$z^{11} = -\frac{4}{3K\Delta}\left[p_{(2)}^2\left(p_{(0)}^2 + p_{(2)}^2 - p_{(1)}^2\right) - p_{(1)}^2 p_{(2)}^2\right] \tag{A.12}$$

$$z^{12} = z^{21} = -\frac{4}{3K\Delta}\left[\left(p_{(0)}^2 + p_{(2)}^2 - p_{(1)}^2\right)\left(p_{(0)}^2 - p_{(2)}^2 + p_{(1)}^2\right) - p_{(1)}^2 p_{(2)}^2\right]$$

$$z^{22} = -\frac{4}{3K\Delta}\left[p_{(1)}^2\left(p_{(0)}^2 - p_{(2)}^2 + p_{(1)}^2\right) - p_{(1)}^2 p_{(2)}^2\right]$$

$$T = (T^{\mu\nu}):\ T^{\mu\nu} = Z^{\nu\mu} \tag{A.13}$$

$$R = (R^{\mu\nu}):\ R^{\mu\nu} = X^{\mu\nu}\big|_{p_{(1)} \leftrightarrow p_{(2)}} \tag{A.14}$$

Remembering (Coleman, 1975)

$$\int \exp\left(\mathrm{A}_{lm}\mathrm{Y}^l\mathrm{Y}^m + \mathrm{V}_l\mathrm{Y}^l\right) dY = \frac{\pi^4}{(\det \mathrm{A})^{1/2}} \exp\left[-\frac{1}{4}(\mathrm{A}^{-1})^{lm}\mathrm{V}_l\mathrm{V}_m\right] \tag{A.15}$$

the vertex is, at last, calculated as in (98).

REFERENCES

Bolza, O. (1904). In *Lectures of the Calculations of Variations*. Chicago, Illinois.

Coleman, S. (1975). *Laws of Hadronic Matter*, "Secret Symmetry Breakdown and Gauge Fields," formula 4.9, Zichichi editor.

Dirac, P. A. M. (1964). In *Lectures on Quantum Mechanics*. Yeshiva University, New York.

Feynman, R. P., and Hibbs, A. R. (1965). In *Quantum Mechanics and Path Integrals*. McGraw-Hill, New York.

Kalb, M., and Van Alstine, P. (1976). In "Invariant Singular Actions for the Relativistic Two Body Problem: A Hamiltonian Formulation." Yale Report C00-3075-146.

Karr, T. J. (1976). In University of Maryland CPT Technical Report 76-085.

Kim, Y. S., and Noz, M. E. (1973). *Physical Review D*, **8**, 3521.

Mandelstam, S. (1973). *Nuclear Physics B*, **64**, 205.

Rosen, G. (1969). In *Formulation of Classical and Quantum Dynamical Theory*. Academic Press, New York.

Schouten, J. A., and Kulk, W.v.d. (1949). In *Pfaff's Problem and its Generalizations*. Clarendon Press, Oxford.

Shanmugadhasan, S. (1973). *Journal of Mathematical Physics*, **14**, 677.

Sudarshan, E. C. G., and Mukunda, N. (1974). In *Classical Dynamics: A Modern Perspective*. John Wiley and Sons.

Von Neumann, J. (1955). In *Mathematical Foundations of Quantum Mechanics*. Princeton University Press, Princeton, New Jersey.